A PROBABILISTIC METHOD OF DETERMINING FATIGUE LIFE OF A STRUCTURAL COMPONENT USING THE PARIS FORMULA AND THE PROBABILITY DENSITY FUNCTION OF TIME OF EXCEEDING THE BOUNDARY CONDITION – AN OUTLINE

Henryk Tomaszek

Air Force Institute of Technology
ul. Księcia Bolesława 6, 01-494 Warszawa, Poland
tel.: +48 22 6852163, fax: +48 22 6852163

Sławomir Klimaszewski

Air Force Institute of Technology
ul. Księcia Bolesława 6, 01-494 Warszawa, Poland
tel.: +48 22 6852113, fax: +48 22 6852163
e-mail: slawomir.klimaszewski@itwl.pl

Mariusz Zieja

Air Force Institute of Technology
ul. Księcia Bolesława 6, 01-494 Warszawa, Poland
tel.: +48 22 6851913, fax: +48 22 6852163
e-mail: mariusz.zieja@itwl.pl

Abstract

An attempt has been made to present a probabilistic method to determine fatigue life of an aeronautical structure's component by means of a density function of time a growing crack needs to reach the boundary condition. It has been assumed that in a component of a structure given consideration there is a small crack that grows due to fatigue load affecting it. After having reached the boundary value the component in question loses its usability. Time of the crack growth up to the boundary value is termed a fatigue life of the component. From the aspect of physics, the propagation of a crack within the component, if approached in a deterministic way, is described with the Paris's relationship for m = 2. To model the fatigue crack growth a difference equation has been applied, from which the Fokker-Planck equation has been derived to be then followed with a density function of the growing crack. The in this way found density function of the crack length has been applied to find density function of time of reaching the boundary condition. This function has been used in the present paper to determine the randomly approached fatigue life of a component of a structure.

The present paper has been prepared for the case there is coefficient m = 2 in the Paris formula. With the in the paper presented scheme, one can find fatigue life of the structure's component for the case $m \neq 2$.

Keywords: *fatigue life, density function, fatigue cracking*

1. Introduction

A matter under consideration is a method to determine fatigue life of a structural component of an aircraft. The following assumptions have been made:

- the component's health/maintenance status has been determined with one parameter only, i.e. the length of a crack therein. The actual value of the parameter has been denoted with *l*,
- any change in the crack length may only occur in the course of the system/device being operated,

- in the case given consideration the Paris formula takes the following form:

$$\frac{dl}{dN_z} = CM_k^m \left(\sigma_{\max}\right)^m \pi^{\frac{m}{2}} l^{\frac{m}{2}}, \tag{1}$$

where:

C, m - material constants,

N_z - a variable that denotes the number of the component-affecting load cycles due to the system's vibration,

M_k - coefficient of the finiteness of the component's dimensions at the crack location,

σ_{max} - maximum load defined with equation (2),

- the load upon the structure's component, with the system's vibration taken into account, is a destructive factor. Let us assume we've got a component-affecting-load spectrum, with account taken of vibration. The spectrum allows for the determination of:
  - the total number of load cycles N_c in the course of one flight assumed a standard cycle,
  - maximum loads within thresholds in the assumed spectrum amount to $\sigma_1^{\max}, \sigma_2^{\max}, \ldots, \sigma_L^{\max}$ (the assumed number of thresholds in the spectrum is L),
  - the number of repetitions of specific threshold values of the loading during one flight (standard load) n_i, where:

$$N_c = \sum_{i=1}^{L} n_i ,$$

- maximum values of loads within the assumed thresholds are found in the following way:

$$\sigma_i^{\max} = \frac{\sigma_i^{\max} + \sigma_i^{\min}}{2} + \sigma_i^a , \tag{2}$$

where:

$\sigma_i^{\max}$ - maximum value of the cyclic load within the i-th threshold,

$\sigma_i^{\min}$ - minimum value of the cyclic load within the i-th threshold,

σ_i^a - the amplitude of the cyclic load within the i-th threshold.

- The following frequencies of the occurrence of loads correspond to values thereof within the thresholds $\sigma_1^{\max}, \sigma_2^{\max}, \ldots, \sigma_L^{\max}$:

$$\frac{n_1}{N_c} = P_1 , \ \frac{n_2}{N_c} = P_2 , \ \ldots, \ \frac{n_L}{N_c} = P_L .$$

2. An outline of the method to determine probability density function of the component's crack length

Relationship (1) may be expressed against the flying time of the aircraft. Therefore, we assume that:

$$N_z = \lambda t , \tag{3}$$

where:

λ - the occurrence rate of load cycles upon the component,

t - flying time of the aircraft.

In the case under consideration:

$$\lambda = \frac{1}{\Delta t} ,$$

where Δt - the average duration of the vibration-attributable fatigue-load cycle.

The relationship (1) against the flying time takes the following form:

$$\frac{dl}{dt} = \lambda C M_k^m \left(\sigma_{\max}\right)^m \pi^{\frac{m}{2}} l^{\frac{m}{2}}. \tag{4}$$

Having applied the hitherto made assumptions, one can proceed to determine the relationship that describes the dynamics of the fatigue-crack growth, i.e. of the increase in its length.

Let $U_{l,t}$ denote the probability that at the time t (for the flying time equal to t) the crack reaches the length l. With the above-shown notation used, the dynamics of the crack length increase can be described with the following difference equation:

$$U_{l,t+\Delta t} = P_1 U_{l-\Delta l_1,t} + P_2 U_{l-\Delta l_2,t} + ... + P_L U_{l-\Delta l_L,t}, \tag{5}$$

where:

P_i - probability that the load $\sigma_i^{\max}$ defined with equation (2) occurs, where $i = 1,2,3,.....,L$ and $P_1 + P_2 + P_3 + ... + P_L = 1$,

Δl_i - crack increment in time Δt for the load equal to $\sigma_i^{\max}$, where $i = 1,2,3,.....,L$. The increments are to be found on the grounds of the dependence (4).

Equation (5) in function notation takes the following form:

$$u_{l,t+\Delta t} = \sum_{i=1}^{L} P_i u\left(l - \Delta l_i, t\right), \tag{6}$$

where $u(l,t)$ - the probability density function of the crack length, which depends on the flying time of the aircraft.

The difference equation (6) can be rearranged in the following partial differential equation of the Fokker-Planck type [3]:

$$\frac{\partial u(l,t)}{\partial t} = -\alpha(t)\frac{\partial u(l,t)}{\partial l} + \frac{1}{2}\beta(t)\frac{\partial^2 u(l,t)}{\partial l^2}. \tag{7}$$

A particular solution of equation (7) is the crack-length density function of the following form:

$$u(l,t) = \frac{1}{\sqrt{2\pi A(t)}} e^{-\frac{(l-B(t))^2}{2A(t)}}, \tag{8}$$

where:

$B(t)$ - an average crack length for the aircraft's flying time t,

$A(t)$ - crack-length variance for the aircraft's flying time t.

Equation (8) for the total flying time takes the form:

$$u(l,t_N) = \frac{1}{\sqrt{2\pi A(t_N)}} e^{-\frac{(l-B(t_N))^2}{2A(t_N)}}, \tag{9}$$

where:

$t_N = \sum_{i=1}^{N} t_i$,

N - the number of flights by the aircraft,

t_i - duration of the i-th flight.

Coefficients $B(t_N)$ and $A(t_N)$ for the material constant m = 2 are solutions of the integrals [3]:

$$B(t_N) = \int_0^t \alpha(t_N)dt = l_0\left(e^{\lambda\overline{C_2}t_N} - 1\right), \tag{11}$$

$$A(t_N) = \int_0^t \beta(t_N)dt = \frac{1}{2} l_0^2 \overline{C_2} \omega \left(e^{2\lambda \overline{C_2} t_N} - 1 \right), \tag{12}$$

where:

$\overline{C_2} = C_2 \mathrm{E}[(\sigma^{\max})^2]$,

$C_2 = C M_k^2 \pi$,

$\omega = \dfrac{\mathrm{E}[(\sigma^{\max})^4]}{(\mathrm{E}[(\sigma^{\max})^2])^2}$.

3. An outline of the method to find the probability density function of time of exceeding the permissible (boundary) value by the length of the crack in the component, for m = 2

Using the density function of the crack length (9) dependant on the flying time of the aircraft, one can determine the probability that the actual length of the crack in the aircraft structure's component exceeds the permissible value within the time interval (0, t_N). The relationship is as follows:

$$Q(t_N, l_d) = \int_{l_d}^{\infty} u(l, t_N) dl \,, \tag{13}$$

where l_d - the permissible value of the crack length as determined for some assumed risk of failure to the structural component.

The probability density function of the flying time up to the moment the crack exceeds the permissible value will be determined by the following equation:

$$f(t) = \frac{\partial}{\partial t_N} Q(t_N, l_d) \,. \tag{14}$$

From equation (14) the following is derived:

$$f(t_N, l_d) = u(l_d, t_N)[l_0 \lambda \overline{C_2} e^{\lambda \overline{C_2} t_N} + \frac{(l_d - l_0 (e^{\lambda \overline{C_2} t_N} - 1) \lambda \overline{C_2} e^{2\lambda \overline{C_2} t_N})}{(e^{2\lambda \overline{C_2} t_N} - 1)}], \tag{15}$$

where:

$$u(l_d, t_N) = \frac{1}{\sqrt{2\pi (\frac{1}{2} l_0^2 \overline{C_2} \omega (e^{2\lambda \overline{C_2} t_N} - 1))}} e^{-\frac{(l_d - l_0 (e^{\lambda \overline{C_2} t_N} - 1))^2}{l_0^2 \overline{C_2} \omega (e^{2\lambda \overline{C_2} t_N} - 1)}} \,. \tag{16}$$

The way of finding the probability density function of time of exceeding the permissible condition (15) is given in [3], pp. 87-90.

4. An outline of the way of estimating life of the aircraft structure's component, with the probability density function of time of exceeding the permissible condition for m = 2

The formula for the reliability of the aircraft structure's component can be written down in the form:

$$R(t_N) = 1 - \int_0^t f(t_N, l_d) dt \,, \tag{17}$$

where the probability density function $f(t_N, l_d)$ is given by the formula (15).

The unreliability of the component is then defined by the equation:

$$Q(t_N)=\int_0^t U(l_d,t_N)[l_0\lambda\overline{C_2}e^{\lambda\overline{C_2}t_N}+\frac{(l_d-l_0(e^{\lambda\overline{C_2}t_N}-1)\lambda\overline{C_2}e^{2\lambda\overline{C_2}t_N})}{(e^{2\lambda\overline{C_2}t_N}-1)}]dt\,, \tag{18}$$

where $u(l_d,t_N)$ is determined with the formula (16).

The integral (18) should be re-arranged in the simpler form and the problem reduced to solving the indefinite integral:

$$\int f(t_N,l_d)dt\,. \tag{19}$$

The following change has been made in the integrand:

$$\underbrace{\frac{(l_d-l_0(e^{\lambda\overline{C_2}t_N}-1))^2}{l_0^2\overline{C_2}\omega(e^{2\lambda\overline{C_2}t_N}-1)}}_{\text{"1"}}=\underbrace{\frac{(l_0(e^{\lambda\overline{C_2}t_N}-1)-l_d)^2}{l_0^2\overline{C_2}\omega(e^{2\lambda\overline{C_2}t_N}-1)}}_{\text{"2"}}\,. \tag{20}$$

Expression "1" is to be replaced with expression "2", and expression "2" is denoted with z:

$$\frac{(l_0(e^{\lambda\overline{C_2}t_N}-1)-l_d)^2}{l_0^2\overline{C_2}\omega(e^{2\lambda\overline{C_2}t_N}-1)}=z\,.$$

Hence,

$$\frac{dz}{dt}=\frac{2[l_0(e^{\lambda\overline{C_2}t_N}-1)-l_d]l_0\lambda\overline{C_2}e^{\lambda\overline{C_2}t_N}[l_0^2\overline{C_2}\omega(e^{2\lambda\overline{C_2}t_N}-1)]-[l_0(e^{\lambda\overline{C_2}t_N}-1)-l_d]^2 2\lambda l_0^2\overline{C_2}^2\omega e^{2\lambda\overline{C_2}t_N}}{l_0^4\overline{C_2}^2\omega^2(e^{2\lambda\overline{C_2}t_N}-1)^2},$$

$$\frac{dz}{dt}=\frac{2l_0^3\overline{C_2}^2\lambda\omega[l_0(e^{\lambda\overline{C_2}t_N}-1)-l_d](e^{2\lambda\overline{C_2}t_N}-1)e^{\lambda\overline{C_2}t_N}-2\lambda l_0^2\overline{C_2}^2\omega[l_0(e^{\lambda\overline{C_2}t_N}-1)-l_d]^2e^{2\lambda\overline{C_2}t_N}}{l_0^4\overline{C_2}^2\omega^2(e^{2\lambda\overline{C_2}t_N}-1)^2},$$

$$\frac{dz}{dt}=\frac{2l_0\lambda[l_0(e^{\lambda\overline{C_2}t_N}-1)-l_d](e^{2\lambda\overline{C_2}t_N}-1)e^{\lambda\overline{C_2}t_N}-2\lambda[l_0(e^{\lambda\overline{C_2}t_N}-1)-l_d]^2e^{2\lambda\overline{C_2}t_N}}{l_0^2\omega\ (e^{2\lambda\overline{C_2}t_N}-1)^2},$$

$$dt=\frac{l_0^2\omega\ (e^{2\lambda\overline{C_2}t_N}-1)^2}{2l_0\lambda[l_0(e^{\lambda\overline{C_2}t_N}-1)-l_d](e^{2\lambda\overline{C_2}t_N}-1)e^{\lambda\overline{C_2}t_N}-2\lambda[l_0(e^{\lambda\overline{C_2}t_N}-1)-l_d]^2e^{2\lambda\overline{C_2}t_N}}dz\,.$$

Then, the substitution has been made in the indefinite integral:

$$\int l_0\lambda\overline{C_2}e^{\lambda\overline{C_2}t_N}+\frac{[(l_d-l_0(e^{\lambda\overline{C_2}t_N}-1)]\lambda\overline{C_2}e^{2\lambda\overline{C_2}t_N}}{(e^{2\lambda\overline{C_2}t_N}-1)}\times$$
$$\times[\frac{l_0^2\omega\ (e^{2\lambda\overline{C_2}t_N}-1)^2}{2l_0\lambda[l_0(e^{\lambda\overline{C_2}t_N}-1)-l_d](e^{2\lambda\overline{C_2}t_N}-1)e^{\lambda\overline{C_2}t_N}-2\lambda[l_0(e^{\lambda\overline{C_2}t_N}-1)-l_d]^2e^{2\lambda\overline{C_2}t_N}}]\times \tag{21}$$
$$\times\frac{1}{\sqrt{\pi l_0^2\overline{C_2}\omega(e^{2\lambda\overline{C_2}t_N}-1))}}e^{-z}dz\,.$$

Hence,

$$\frac{1}{2\sqrt{\pi}}\int[l_0\lambda e^{\lambda\overline{C_2}t_N}(e^{2\lambda\overline{C_2}t_N}-1)+(l_d-l_0(e^{\lambda\overline{C_2}t_N}-1)]\lambda e^{2\lambda\overline{C_2}t_N}]\times$$
$$\times[\frac{1}{l_0\lambda(e^{2\lambda\overline{C_2}t_N}-1)e^{\lambda\overline{C_2}t_N}+[\lambda(l_d-l_0(e^{\lambda\overline{C_2}t_N}-1)e^{2\lambda\overline{C_2}t_N}]}]\frac{1}{\sqrt{z}}e^{-z}dz\,.$$

Therefore, the following is arrived at:

$$\frac{1}{2\sqrt{\pi}}\int\frac{1}{\sqrt{z}}e^{-z}dz\,.$$

After the rearrangements the indefinite integral (21) takes the following form:

$$\frac{1}{2\sqrt{\pi}}\int\frac{1}{\sqrt{z}}e^{-z}dz\,. \tag{22}$$

Then, the second substitution has to be made in the integral (22), which should take the form:

$$\sqrt{z}=w\,,$$
$$\frac{dw}{dz}=\frac{1}{2\sqrt{z}}\,,$$
$$\frac{dz}{dw}=2w\,,$$
$$dz=2wdw\,. \tag{23}$$

The dependence (23) is inserted in the integral (22). Hence, the following is arrived at:

$$\frac{1}{2\sqrt{\pi}}\int\frac{1}{w}e^{-w^2}\,2wdw=\frac{1}{\sqrt{\pi}}\int e^{-w^2}dw\,. \tag{24}$$

One more substitution:

$$w^2=\frac{y^2}{2}\,,$$
$$2wdw=ydy\,,$$
$$dw=\frac{y}{2w}dy\,,$$
$$dw=\frac{dy}{\sqrt{2}}\,. \tag{25}$$

Hence, after inserting (25) in (24) the following integral is effected:

$$\frac{1}{\sqrt{2\pi}}\int e^{-\frac{y^2}{2}}dy\,, \tag{26}$$

where y takes value determined with the dependence (27), since

$$\frac{y^2}{2}=w^2,\qquad w=\sqrt{z},\qquad z=\frac{(l_0(e^{\lambda\overline{C_2}t_N}-1)-l_d)^2}{l_0^2\overline{C_2}\omega(e^{2\lambda\overline{C_2}t_N}-1)}\,,$$
$$w=\sqrt{\frac{(l_0(e^{\lambda\overline{C_2}t_N}-1)-l_d)^2}{l_0^2\overline{C_2}\omega(e^{2\lambda\overline{C_2}t_N}-1)}}\,,$$
$$w=\frac{(l_0(e^{\lambda\overline{C_2}t_N}-1)-l_d)}{\sqrt{l_0^2\overline{C_2}\omega(e^{2\lambda\overline{C_2}t_N}-1)}}\,,$$
$$w^2=\frac{y^2}{2}\,,$$

$$y^2 = 2w^2,$$

$$y = \sqrt{2w^2},$$

$$y = w\sqrt{2} = \sqrt{2}\frac{(l_0(e^{\lambda\overline{C_2}t_N} - 1) - l_d)}{\sqrt{l_0^2\overline{C_2}\omega(e^{2\lambda\overline{C_2}t_N} - 1)}}. \tag{27}$$

Having inserted the results gained in the equation (17) and remembering about a suitable notation of the limits of integration, the following dependence for the reliability is arrived at:

$$R(t) = 1 - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y(t)} e^{-\frac{y^2}{2}}dy, \tag{28}$$

where equation (27) should be substituted for the upper limit of the integral $y(t)$.

The cumulative distribution function for the standard Gaussian (normal) distribution takes the form:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}dy.$$

With the above-shown dependence taken into account, the formula for the reliability of the structure's component is expressed with the following equation:

$$R(t_N) = 1 - \Phi(\sqrt{2}\frac{(l_0(e^{\lambda\overline{C_2}t_N} - 1) - l_d)}{\sqrt{l_0^2\overline{C_2}\omega(e^{2\lambda\overline{C_2}t_N} - 1)}}). \tag{29}$$

Hence, reliability of the structure's component will be determined with the following dependence:

$$Q(t_N) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\sqrt{2}\frac{[l_0(e^{\lambda\overline{C_2}t_N} - 1) - l_d]}{\sqrt{l_0^2\overline{C_2}\omega(e^{2\lambda\overline{C_2}t_N} - 1)}}} e^{-\frac{y^2}{2}}dy. \tag{30}$$

Having found (assumed) the level of risk of a failure to the structure's component, i.e. the level of exceeding the permissible value of the length of a crack in this component, we get:

$$Q(t_N) = Q^*. \tag{31}$$

Hence,

$$Q^* = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\vartheta} e^{-\frac{y^2}{2}}dy. \tag{32}$$

For the assumed value of Q^*, the value of the upper limit of the integral (for which the integral on the right side of the equation (32) takes value Q^*) is to be found in the standard Gaussian distribution tables.

Hence, the following dependence is arrived at:

$$\vartheta = \sqrt{2}\frac{(l_0(e^{\lambda\overline{C_2}t_N} - 1) - l_d)}{\sqrt{l_0^2\overline{C_2}\omega(e^{2\lambda\overline{C_2}t_N} - 1)}}, \tag{33}$$

$$\frac{\vartheta}{\sqrt{2}} = \frac{(l_0(e^{\lambda\overline{C_2}t_N} - 1) - l_d)}{\sqrt{l_0^2\overline{C_2}\omega(e^{2\lambda\overline{C_2}t_N} - 1)}}.$$

We assume that

$$\frac{\vartheta}{\sqrt{2}} = \vartheta^* .$$

Hence,

$$\vartheta^* = \frac{(l_0(e^{\lambda \overline{C_2} t_N} - 1) - l_d)}{\sqrt{l_0^2 \overline{C_2} \omega (e^{2\lambda \overline{C_2} t_N} - 1)}} . \tag{34}$$

From (34) we can find time t_N^*, for which the equality relation (34) takes place. Time t_N^* will be the searched life of the structure's component, i.e. it will be the aircraft's flying time for the assumed risk of exceeding the permissible value of the crack length. We assume that:

$$e^{\lambda \overline{C_2} t_N} = x . \tag{35}$$

Hence,

$$\vartheta^* = \frac{(l_0(x-1) - l_d)}{\sqrt{l_0^2 \overline{C_2} \omega (x^2 - 1)}} . \tag{36}$$

From (36) we can find x. With some specific value of x gained from the dependence (35), we can find t_N^*:

$$\begin{aligned} e^{\lambda \overline{C_2} t_N} &= x , \\ \lambda \overline{C_2} t_N &= \ln x , \\ t_N^* &= \frac{\ln x}{\lambda \overline{C_2}} . \end{aligned} \tag{37}$$

Formula (37) determines fatigue life of the aircraft structure's component t_N^* for the assumed risk of exceeding the boundary condition Q^*.

5. Final remarks

What has been presented in the paper is an outline of a method to determine fatigue life of an aircraft structure's component. What provokes a fatigue process is a random load in the form of load spectrum. It should be emphasised that it is possible to find fatigue life of a component using a more complex load spectrum. It has been assumed in the paper that the sequence of load cycles, as far as values thereof are concerned remains of no effect upon the crack growth rate. All the dependences arrived at enable specific calculations, if we have values of material constants and data on the load spectrum.

The present paper has been prepared for the case there is coefficient m = 2 in the Paris formula. With the in the paper presented scheme, one can find fatigue life of the structure's component for the case $m \neq 2$.

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